EFFECT OF RHEOLOGICAL FACTORS ON THE STATE OF STRESS OF A BENT MULTILAYER BEAM

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Inzhenerno-Fizicheskii Zhurnal, Vol. 12, No. 6, pp. 769–776, 1967

UDC 532.135

The author examines the problem of the effect of the time factor on the state of stress of a two-layer cantilever beam composed of orthotropic strips of the same thickness but with different elastic properties. The problem is solved using the mathematical apparatus of the theory of Volterra-type linear integral operators.

Using the equations representing the instantaneous-elastic state of the beam [1] and the Volterra principle, we obtain the following expressions for the stress:

$$\bar{\sigma}_x^{(1)} = \frac{6\bar{E}_1^{(1)}}{h\bar{S}}(M - Px)(2\bar{S}_1 y - \bar{S}_2), \tag{1}$$

$$\bar{\sigma}_x^{(2)} = \frac{\bar{E}_1^{(2)}}{\bar{E}_1^{(1)}}\bar{\sigma}_x^{(1)}. \tag{2}$$

Here

$$\bar{E}_1^{(k)} = E_{10}^{(k)}\left[1 - \chi^{(k)} Q^*(-\beta^{(k)})\right], \quad k = 1,\ 2;$$

$$\bar{S}_1 = b_1\bar{E}_1^{(1)} + (b - b_1)\bar{E}_1^{(2)};$$

$$\bar{S}_2 = b_1^2\bar{E}_1^{(1)} + (b^2 - b_1^2)\bar{E}_1^{(2)};$$

$$\bar{S} = b_1^4(\bar{E}_1^{(1)})^2 + 2b_1(b - b_1)(2b^2 - bb_1 + b_1^2)\times$$
$$\times\bar{E}_1^{(2)}\bar{E}_1^{(1)} + (b - b_1)^4(\bar{E}_1^{(2)})^2;$$

$\chi^{(k)}$, $\beta^{(k)}$ are the rheological parameters, h the thickness of the layers, b the total depth of the beam, b_1 the depth of the first layer, M the bending moment, P the shear force, $E_{10}^{(k)}$ $(k = 1,\ 2)$ the elastic moduli of the layers, and $Q^*(-\beta^{(k)})$ the Volterra-type integral operators.

The problem consists in evaluating the operator expressions (1) and (2).

We rewrite Eq. (1) in the form:

$$\bar{\sigma}_x^{(1)} = \frac{K}{L} = 6(M - Px)b_1^4\{(2yb_1 - b_1^2)(\bar{E}_1^{(1)})^2 +$$
$$+ [2y(b - b_1) + (b^2 - b_1^2)]\,\bar{E}_1^{(1)}\bar{E}_1^{(2)}\}\Big/ h\Big[(\bar{E}_1^{(1)})^2 +$$
$$+ \frac{2(b - b_1)}{b_1^3}(2b^2 - bb_1 + b_1^2)\times$$
$$\times\bar{E}_1^{(2)}\bar{E}_1^{(1)} + \frac{(b - b_1)^4}{b_1^4}(\bar{E}_1^{(2)})^2\Big]. \tag{3}$$

It will be necessary to carry out the inversion of the quadratic trinomial in the denominator.

In particular, let $b^2 + b_1^2 - bb_1 = 0$. The denominator of expression (3) then assumes the form

$$h\left[\bar{E}_1^{(1)} + \left(\frac{b - b_1}{b_1}\right)^2\bar{E}_1^{(2)}\right]^2 \tag{4}$$

and after transformations

$$h\left[E_{10}^{(1)} + \left(\frac{b - b_1}{b_1}\right)^2 E_{10}^{(2)}\right]^2\times$$
$$\times\left[1 - \chi_1 Q^*(-\beta^{(1)}) - \chi_2 Q^*(-\beta^{(2)})\right]^2. \tag{5}$$

Here,

$$\chi_1 = \frac{\chi^{(1)}E_{10}^{(1)}b_1^2}{b_1^2E_{10}^{(1)} + (b - b_1)^2E_{10}^{(2)}};$$

$$\chi_2 = \frac{(b - b_1)^2\chi^{(2)}E_{10}^{(2)}}{b_1^2E_{10}^{(1)} + (b - b_1)^2E_{10}^{(2)}}.$$

The known properties of Q* operators are insufficient for the inversion of expression (5). Therefore we will prove the following important property of Volterra-type integral operators, which is required to solve many problems of the theory of elastic media with memory effects:

$$\bar{W}_n = f(t)/\left[1 + \eta_1 Q^*(-\beta_1) + \eta_2 Q^*(-\beta_2)\right]^n =$$
$$= \left[1 + \zeta_1 Q^*(r_1) + \zeta_2 Q^*(r_2)\right]^n f(t), \tag{6}$$

where the parameters ζ_i and r_i are determined in the usual way [2]. For n = 2 we have

$$\bar{W}_2\left[1 + \eta_1 Q^*(-\beta_1) + \eta_2 Q^*(-\beta_2)\right] =$$
$$= f(t)/\left[1 + \eta_1 Q^*(-\beta_1) + \eta_2 Q^*(-\beta_2)\right] =$$
$$= \left[1 + \zeta_1 Q^*(r_1) + \zeta_2 Q^*(r_2)\right]f(t).$$

Consequently,

$$\bar{W}_2 = f(t) + 2\zeta_1\int_0^t Q(r_1,\ t - \tau)f(\tau)\,d\tau +$$
$$+ 2\zeta_2\int_0^t Q(r_2,\ t - \tau)f(\tau)\,d\tau +$$
$$+ \zeta_1^2\int_0^t Q(r_1,\ t - \tau)\int_0^\tau Q(r_1,\ \tau - s)f(s)\,ds\,d\tau +$$
$$+ \zeta_1\zeta_2\int_0^t Q(r_2,\ t - \tau)\int_0^\tau Q(r_1,\ \tau - s)f(s)\,ds\,d\tau +$$
$$+ \zeta_2\zeta_1\int_0^t Q(r_1,\ t - \tau)\int_0^\tau Q(r_2,\ \tau - s)f(s)\,ds\,d\tau +$$
$$+ \zeta_2^2\int_0^t Q(r_2,\ t - \tau)\int_0^\tau Q(r_2,\ \tau - s)f(s)\,ds\,d\tau.$$

Using Dirichlet's theorem on the order of integration, we then obtain

$$\overline{W}_2 = f(t) + 2\zeta_1 \int_0^t Q(r_1, t-\tau) f(\tau)\, d\tau + 2\zeta_2 \int_0^t Q(r_2, t-\tau) f(\tau)\, d\tau + \zeta_1^2 \int_0^t f(s)\, ds \times \int_s^t Q(r_1 t-\tau) Q(r_1, \tau - s)\, d\tau + \zeta_2^2 \int_0^t f(s)\, ds \times \int_s^t Q(r_2, t-\tau) Q(r_2, \tau - s)\, d\tau + 2\zeta_1\zeta_2 \times \int_0^t f(s)\, ds \int_s^t Q(r_1, t-\tau) Q(r_2, \tau - s)\, d\tau.$$

Considering that the last three terms are respectively the squares of the operators $Q^*(r_1)$, $Q^*(r_2)$ and the product of these operators, it is clear that Eq. (6) is valid for $n = 2$. Then, by mathematical induction it is easy to show that property (6) is valid for any n.

This property makes it possible to invert the operator expression in the denominator of (3) in the particular case when $b^2 + b_1^2 - bb_1 = 0$ and hence in the last analysis evaluate the expression for $\overline{\sigma}_x$.

We now turn to the general case when $b^2 + b_1^2 - bb_1 \neq 0$. We convert the denominator of (3) to the form

$$h(\overline{E}_1^{(2)})^2 (\overline{E}^2 + 2k\overline{E} + m) = h(\overline{E}_1^{(2)})^2 (\overline{E} - \gamma)(\overline{E} - \delta). \tag{7}$$

Here, γ and δ are the roots of the trinomial in the parentheses;

$$k = \frac{2(b - b_1)}{b_1^3}(2b^2 - bb_1 + b_1^2); \quad \overline{E} = \frac{\overline{E}_1^{(1)}}{\overline{E}_1^{(2)}}.$$

We will now consider three cases:

1. The roots γ and δ are real and equal. This case is analogous to the particular case considered above when $b^2 + b_1^2 - bb_1 = 0$. To invert the operator expression we use property (6).

2. The roots γ and δ are real and different. After reducing to prime factors, from (3) we obtain

$$\overline{\sigma}_x^{(1)} = \frac{\overline{K}}{h\overline{E}_1^{(2)}(\gamma - \delta)} \times \left(\frac{1}{\overline{E}_1^{(1)} - \gamma \overline{E}_1^{(2)}} - \frac{1}{\overline{E}_1^{(1)} - \delta \overline{E}_1^{(2)}} \right), \tag{8}$$

and after further transformations,

$$\overline{\sigma}_x^{(1)} = [\overline{K} h \overline{E}^{(2)} (\gamma - \delta)] \{ [(\overline{E}_{10}^{(1)} - \gamma E_{10}^{(2)}) [1 - \lambda_1 Q^*(-\beta^{(1)}) - \lambda_2 Q^*(-\beta^{(2)})]]^{-1} - [(E_{10}^{(1)} - \delta E_{10}^{(2)}) [1 - \lambda_1' Q^*(-\beta^{(1)}) - \lambda_2' Q^*(-\beta^{(2)})]]^{-1} \}. \tag{9}$$

Here,

$$\lambda_k = (-1)^{k-1} \frac{d_k E_{10}^{(k)} \chi^{(k)}}{E_{10}^{(1)} - \gamma E_{10}^{(2)}}; \quad \lambda_k' = \frac{d_k' E_{10}^{(k)} \chi^{(k)}}{E_{10}^{(1)} - \delta E_{10}^{(2)}};$$

$$d_1 = d_1' = 1; \quad d_2 = \gamma; \quad d_2' = \delta.$$

After inversion of the denominator, we obtain [2]

$$\overline{\sigma}_x^{(1)} = \frac{\overline{K}}{h(\gamma - \delta) E_{10}^{(2)}} [1 + \chi^{(2)} Q^*(\chi^{(2)} - \beta^{(2)})] \times \left\{ \frac{1}{E_{10}^{(1)} - \gamma E_{10}^{(2)}} [1 + a_1 Q^*(r_1) + a_2 Q^*(r_2)] - \frac{1}{E_{10}^{(1)} - \delta E_{10}^{(2)}} [1 + a_1' Q^*(r_1') + a_2' Q^*(r_2')] \right\}. \tag{10}$$

Following the necessary transformations we obtain

$$\overline{\sigma}_x^{(1)} = \frac{\overline{K}}{L_0} \left[1 + \sum_{i=1}^{5} \gamma_i Q^*(r_i) \right]. \tag{11}$$

Here, L_0 is the instantaneous value of the operator $\overline{L}$;

$$r_3 = \chi^{(2)} - \beta^{(2)}; \quad r_4 = r_1'; \quad r_5 = r_2';$$

$$\gamma_i = (E_0 - \gamma)(E_0 - \delta) E_{10}^{(2)} / (E_{10}^{(1)} - d_i E_{10}^{(2)})(\gamma - \delta)\, \xi_i l_i$$

$$(i = 1, 2, 4, 5; \quad d_1 = d_2 = \gamma; \quad d_3 = d_4 = \delta; \quad l_1 = l_2 = 1; \quad l_3 = l_4 = -1);$$

$$\gamma_3 = [\xi_3 (E_{10}^{(1)} - \delta E_{10}^{(2)}) + \xi_6 (E_{10}^{(1)} - \gamma E_{10}^{(2)})] / E_{10}^{(1)} - \delta E_{10}^{(2)} - E_{10}^{(1)} + \gamma E_{10}^{(2)});$$

$$\xi_i = a_i - \frac{\chi^{(2)} a_i}{\chi^{(2)} - \beta^{(2)} - r_i} \quad (i = 1, 2);$$

$$\xi_3 = 1 + \sum_{k=1}^{2} \frac{a_k}{\chi^{(2)} - \beta^{(2)} - r_k}. \tag{12}$$

The quantities ξ_4, ξ_5, ξ_6 are obtained from (12) by substituting a_1' and a_2' for the coefficients a_1, a_2.

Finally, we obtain

$$\overline{\sigma}_x^{(1)} = \sigma_{x_0}^{(1)} \left[1 + \sum_{i=1}^{5} \gamma_i Q^*(r_i) \times \left(1 + \sum_{k=1}^{2} \frac{s_k}{\beta^{(k)} + r_i} + \frac{s_3}{(\beta^{(1)} + r_i)^2} \right) + \sum_{k=1}^{2} s_k Q^*(-\beta^{(k)}) \left(1 - \sum_{i=1}^{5} \frac{\gamma_i}{\beta^{(k)} + r_i} \right) + s_3 Q^{*2}(-\beta^{(1)}) \left(1 - \sum_{i=1}^{5} \frac{\gamma_i}{\beta^{(1)} + r_i} \right) - s_3 Q^*(-\beta^{(1)}) \sum_{i=1}^{5} \frac{\gamma_i}{(\beta^{(1)} + r_i)^2} \right]. \tag{13}$$

Here, $\sigma_{x_0}^{(1)}$ is the instantaneous value of the stress $\sigma_x^{(1)}$.

The coefficients s_i ($i = 1, 2, 3$) in (13) were obtained by evaluating the operator $\overline{K}$ using the fundamental properties of Q^* operators.

3. The roots γ and δ are complex. In this case it is necessary to evaluate an expression of the form:

$$(\bar{E}^2 + 2k\bar{E} + m)^{-1}, \tag{14}$$

which, as may easily be seen, leads to the evaluation of the expression

$$\frac{1}{Q^{*2}(-\beta) + 2pQ^*(-\beta) + q}, \tag{15}$$

where

$$p = \frac{E_0^2 + kE_0}{E_0^2\chi}; \quad q = \frac{E_0^2 + 2kE_0 + m}{E_0^2\chi^2}.$$

In order to obtain this evaluation we will prove the following property of integral operators.

If the operator $Q^*(\lambda)$ is expanded in series as

$$Q^*(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n, \tag{16}$$

then

$$Q^*(-i\chi - \beta) = Q_1^*(\chi, \beta) - iQ_2^*(\chi, \beta),$$
$$Q^*(i\chi - \beta) = Q_1^*(\chi, \beta) + iQ_2^*(\chi, \beta), \tag{17}$$

and

$$Q_1^*(-\beta; -\chi) = Q_1^*(-\beta; \chi);$$
$$Q_2^*(-\beta; -\chi) = -Q_2^*(-\beta; \chi).$$

We note that if $Q^*(\lambda)$ is an operator-resolvent of Volterra type, then the expansion (16) holds, the series on the right side of the expression converging uniformly for $-\infty < \lambda < \infty$.

Using (16), we obtain, in particular,

$$Q^*(-i\chi - \beta) = \sum_{k=0}^{\infty} a_k\beta^k + \sum_{m=1}^{\infty}(-1)^m a_{2m}\chi^{2m} +$$
$$+ \sum_{m=1}^{\infty}\sum_{k=2m+1}^{\infty}(-1)^{k+m} a_k \frac{k(k-1)\ldots(2m+1)}{(k-2m)!} \times$$
$$\times \chi^{2m}\beta^{k-2m} - i\left(a_1\chi - 2a_2\chi\beta + \right.$$
$$\left. + \sum_{l=1}^{\infty}\sum_{k=l}^{\infty}(-1)^{k-l} a_k \chi^{2l-1}\beta^{k-2l+1}\frac{k(k-1)\ldots 2l}{(k-2l+1)!}\right).$$

Having obtained the analogous expression for the operator $Q^*(-i\chi + \beta)$, but with positive imaginary part, we see that expression (17) is valid.

We now proceed with the evaluation of expression (15), writing it in the form

$$1/[Q^{*2}(-\beta) + 2pQ^*(-\beta) + q] =$$
$$= 1/[Q^*(-\beta) - \gamma][Q^*(-\beta) - \delta], \tag{18}$$

where $\gamma = -p + iR$; $\delta = -p - iR$; $R = \sqrt{q - p^2}$.

After reducing to prime factors, we have

$$\frac{1}{Q^{*2}(-\beta) + 2pQ^*(-\beta) + q} = \frac{1}{2iR}\left\{\left(\frac{1}{\delta} - \frac{1}{\gamma}\right) + \right.$$
$$\left. + \left[\frac{1}{\delta^2}Q^*\left(\frac{1}{\delta} - \beta\right) - \frac{1}{\gamma^2}Q^*\left(\frac{1}{\gamma} - \beta\right)\right]\right\}. \tag{19}$$

Transforming (19) with allowance for properties (17), we finally obtain

$$\frac{1}{Q^{*2}(-\beta) + 2pQ^*(-\beta) + q} = \frac{1}{q} +$$
$$+ \frac{p^2 - R^2}{R(p^2 + R^2)^2} Q_2^*\left(\frac{R}{p^2 + R^2}, \beta + \frac{p}{p^2 + R^2}\right) -$$
$$- \frac{2p}{(p^2 + R^2)^2} Q_1^*\left(\frac{R}{p^2 + R^2}, \beta + \frac{p}{p^2 + R^2}\right). \tag{20}$$

As an illustration, we will consider the bending of a two-layer beam consisting of layers of sandstone and argillite. From the experimental creep curves for sandstone and argillite [3], we obtained the following rheological parameters: for sandstone $\chi = 0.001124$ $\sec^{-0.5}$, $\beta = 0.002435$ $\sec^{-0.5}$; for argillite $\chi = 0.00407$ $\sec^{-0.5}$, $\beta = 0.00584$ $\sec^{-0.5}$. It was also assumed that $E_{10}^{(1)} = 3 \cdot 10^{10}$ N/m², $E_{10}^{(2)} = 1.34 \cdot 10^{10}$ N/m², where $E_{10}^{(1)}$ is the instantaneous modulus of elasticity of sandstone and $E_{10}^{(2)}$ that of argillite. To simplify the calculations we took: $6(M - Px) = 1$; $b = 2b_1 = 2$; $y = 1$; $h = 1$. The roots of the quadratic trinomial in the denominator obtained under these assumptions are real and different: $\gamma = -0.08$, $\delta = -13.92$.

As the $Q^*(-\beta)$ operator we took Rabotnov's operator $\ni_\alpha^*(-\beta)$ [4]. In the calculations we employed the following representation of the $\ni_\alpha^*$ operator (at $\alpha = 0.5$) [5]:

$$\ni_\alpha^*(-\beta)\cdot 1 = \frac{1}{\beta}\{1 - E_{1+\alpha}(-\beta t^{1+\alpha})\} =$$
$$= \frac{1}{\beta}\{1 - \exp(\beta t^{(1+\alpha)2})[1 - \Phi(\beta t^{1+\alpha})]\}. \tag{21}$$

Here, $\Phi(\beta t^{1+\alpha})$ is the error function.

The results obtained are presented in the table, which shows that the sign of the stress $\bar{\sigma}_x^{(1)}$ changes at a certain instant of time.

We will now consider the particular case when one of the media is purely elastic; let, for example, $\bar{E}^{(1)} = E_0^{(1)} = \text{const}$. Then the expression for the stress $\bar{\sigma}_x^{(1)}$ takes the form

$$\bar{\sigma}_x^{(1)} = \bar{K}/L_0[1 + \chi_1 Q^*(-\beta) + \chi_2 Q^{*2}(-\beta)]. \tag{22}$$

Here,

$$\chi_1 = -2h\chi\left[kE_0^{(1)}E_0^{(2)} + m(E_0^{(2)})^2\right]/L_0;$$
$$\chi_2 = mh\chi^2(E_0^{(2)})^2/L_0.$$

Reduction to prime factors and subsequent transformations give

$$\bar{\sigma}_x^{(1)} = \frac{\bar{K}}{L_0\gamma\delta}\left[1 + \xi_1 Q^*(-\beta_1) + \xi_2 Q^*(-\beta_2)\right], \tag{23}$$

where

$$\xi_1 = \frac{\gamma}{\delta(\gamma - \delta)}; \quad \xi_2 = -\frac{\delta}{(\gamma - \delta)};$$

$$\beta_i = \beta - \frac{1}{\varkappa_i}; \quad (i = 1, 2); \quad \varkappa_1 = \gamma; \quad \varkappa_2 = \alpha.$$

Variation in Time of the Relative Stress of a Bent Two-Layer Beam Composed of Layers of Sandstone and Argillite

$\sqrt{t}$, $sec^{0.5}$	100	200	300	400	500	600	1000	2000	∞
$\frac{\sigma_0-\sigma_t}{\sigma_0}$	0.89	1.47	1.855	2.12	2.325	2.414	2.763	2.879	3.05

Finally, we obtain

$$\bar{\sigma}_x^{(1)} = \frac{\sigma_{x_0}^{(1)}}{\gamma\delta}\left[1+\xi_1 Q^*(-\beta_1)\times\right.$$

$$\times\left(1+\frac{s}{\beta_1-\beta}\right)+\xi_2 Q^*(-\beta_2)\times$$

$$\left.\times\left(1+\frac{s}{\beta_2-\beta}\right)-s\sum_{k=1}^{2}\frac{\xi_k}{\beta_2-\beta}Q^*(-\beta)\right],$$

$$s=\frac{B_2E_0^{(2)}\chi}{B_1+B_2E_0^{(2)}};\ B_1=(2yb_1-b_1^2)(E_0^{(1)})^2;$$

$$B_2=\left[2y(b-b_1)+(b^2-b_1)^2\right]E_0^{(1)}. \qquad (24)$$

The roots of the quadratic trinomial in the denominator are again real and different: $\gamma = 0.003495$, $\delta = 0.000125$. After inverting the operator expression in the denominator we find that β_1 and β_2 are negative, which should apparently lead to instability of the system in time. However, this does not happen since the expression for $\bar{\sigma}_x$ contains a combination of the operators $Q^*(-\beta_1)$ and $Q^*(-\sigma_2)$, which as $t \to \infty$ gives an indeterminacy of the type $\infty - \infty$ that can be evaluated by L'Hospital's rule. Thus, for the limiting value of the stress we obtain

$$\frac{\sigma_\infty\gamma\delta-\delta_0}{\sigma_0}=0.969.$$

Accordingly, neglecting the rheological properties of one of the components of the beam investigated, under the conditions defined in [1], results in a change in the limiting bending stress. As compared with its original value, the limiting bending stress is greater than when the memory properties of both layers of the beam are taken into account.

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18 October 1966

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